Sommerfeld's formula and uniqueness for the boundary value contact problems

1998 J. Phys. A: Math. Gen. 31 L405

(http://iopscience.iop.org/0305-4470/31/22/001)

J. Phys. A: Math. Gen. **31** (1998) L405–L411. Printed in the UK PII: S0305-4470(98)92936-5

LETTER TO THE EDITOR

Sommerfeld's formula and uniqueness for the boundary value contact problems

I V Andronov† and B P Belinskiy‡

† Scientific Research Institute in Physics, The University of St Petersburg, 1-1 Ulianovskaya Str., 198904 St Petersburg, Russia

‡ Department of Mathematics, The University of Tennessee at Chattanooga, 615 McCallie Avenue, Chattanooga, TN 37403-2598, USA

Received 1 April 1998

Abstract. The expression of the acoustic field scattered on an infinite elastic plate with an arbitrary compact inhomogeneity in terms of the analytic continuation of its scattering diagram is found. This formula allows the uniqueness of the solution for the scattering problem to be proved. The connection of the formula with the Rayleigh hypothesis is discussed.

1. Introduction

This letter deals with the two-dimensional problem of acoustic scattering by an infinite elastic plate with a finite inhomogeneity. A representation for the scattered field is found in terms of the scattering diagram. This formula is used for the proof of the uniqueness of the solution for the scattering problem.

We assume for the above model that there is no absorption either in the fluid or in the plate. In the case of absorption, the uniqueness of the solution can easily be proved by using the second Green formula. However, we do not discuss this elementary case here.

The known results for uniqueness in the problems when no absorption is present are based on explicit representation for the solution. The closed form solution for the scattering problem can be constructed for some specific inhomogeneities only, such as pointwise cracks, supported ribs, etc [1, 2]. The solution is constructed as follows. First, a general solution is found which satisfies all of the equations in the domain and on the boundary except for the conditions describing a particular pointwise inhomogeneity. (According to [3], those conditions are said to be contact and the scattering problems are said to be boundary-contact value problems, (BCVP).) The general solution contains some arbitrary constants, and the contact conditions lead to a linear algebraic system for those constants. The solvability of these systems is equivalent to the uniqueness of the solution for BCVP. The proof of solvability is given in [4]. It is based on the existence of the explicit representation for the general solution. For many other models, the scattering problem is reduced to a Fredholm type equation (see, for example, [5–7]). The complete justification of any numerical algorithm for a Fredholm equation requires the proof of the existence. The Fredholm alternative [8] implies existence if the uniqueness is proved. Hence the proof of the uniqueness theorem for BCVP in an arbitrary domain is of independent interest. The absence of an explicit representation for the general form of the scattered field does not allow the proof of uniqueness [5] to be used. Yet, the partial result is known, namely the solution of the homogeneous BCVP does not contain the scattering diagram and the surface waves [4]. The following question appears. Does a solution of BCVP exist with no components which transfer the energy? All of these questions can be answered by using the direct representation for the scattered field in terms of the scattering diagram and its analytic continuation with respect to the angle of observation. This representation is known for the problem of scattering by a compact obstacle as Sommerfeld's formula [9, 10].

This paper is organized as follows. In section 2 some auxiliary formulae for Green functions are discussed. In section 3 the scattering on an elastic plate with an arbitrary compact inhomogeneity (BCVP) is considered. The representation for the solution of BCVP is derived by means of the Green second formula. The asymptotic expansion of this formula allows us to find the desired Sommerfeld's formula, i.e. the representation of the scattered field in terms of the scattering diagram and its analytic continuation. This continuation is known to be a meromorphic function of the angle on the complex plane [11]. In section 4 the uniqueness of the solution for a particular BCVP is proved by means of Sommerfeld's formula and the analytic properties of the scattering diagram. In section 5 other applications of Sommerfeld's formula, including the connection with the Rayleigh hypothesis, are briefly discussed. Some derivations of this paper are based on the results from [11], similar details are not repeated.

2. Some properties of Green functions

Two auxiliary functions are needed in section 3. Their explicit representation is well known (see [3]). They are Green functions for harmonic vibrations of an acoustic medium $\mathbb{R}^2_+ = \{-\infty < x < +\infty, y > 0\}$ in the presence of an elastic plate $\{y = 0\}$ (Kirchhoff's model is used and the time factor $e^{i\omega t}$ is dropped throughout the paper)

$$G(\boldsymbol{r}, \boldsymbol{r_0}) = -\frac{1}{4\pi} \int \frac{1}{\gamma(\lambda)} e^{i\lambda(x-x_0)} \left(e^{-\gamma(\lambda)|y-y_0|} + \frac{l_0(\lambda)}{l(\lambda)} e^{-\gamma(\lambda)(y+y_0)} \right) d\lambda \tag{2.1}$$

and surface Green function for the same model

$$g(\boldsymbol{r}, x_0) = -\frac{\rho\omega^2}{2\pi} \int \frac{1}{l(\lambda)} e^{i\lambda(x-x_0)-\gamma(\lambda)y} \, d\lambda. \tag{2.2}$$

The Green function is the solution of the Helmholtz equation in a half-plane $\mathbb{R}^2_+$

$$(\Delta_2 + k^2) G(\boldsymbol{r}, \boldsymbol{r_0}) = \delta(\boldsymbol{r} - \boldsymbol{r_0}) \qquad \boldsymbol{r}, \boldsymbol{r_0} \in \mathbb{R}^2_+ \tag{2.3}$$

subject to the boundary condition

$$\left(D \frac{\partial^4}{\partial x^4} - m\omega^2 \right) \frac{\partial G}{\partial y}(x, 0) + \rho\omega^2 G(x, 0) = 0 \qquad -\infty < x < \infty, \ y = 0. \tag{2.4}$$

The surface Green function is the solution of the homogeneous equation (2.3) subject to the boundary condition (2.4) with $\delta(x - x_0)$ instead of zero on the right-hand side. Throughout $\boldsymbol{r} = (x, y)$, $\boldsymbol{r_0} = (x_0, y_0)$, k is the wavenumber of the acoustic medium, D is the cylindrical rigidity of the plate, m is its density, ω is the frequency of oscillations, ρ is the density of the acoustic medium. The radiation conditions are taken in the form of the asymptotic representations

$$G(\boldsymbol{r}, \boldsymbol{r_0}) = \sqrt{\frac{2\pi}{kr}} e^{i(kr - \frac{\pi}{4})} \Psi_G(\varphi, \boldsymbol{r_0}) + o(r^{-1/2}) \qquad \text{as } r = \sqrt{x^2 + y^2} \to \infty \tag{2.5a}$$

uniformly with respect to the polar angle $\varphi = \cos^{-1}(y/r) \in [0, \pi]$,

$$G(\boldsymbol{r}, \boldsymbol{r_0}) = A_G^{\pm}(\boldsymbol{r_0}) \mathrm{e}^{\pm \mathrm{i}\kappa x - \sqrt{\kappa^2 - k^2} y} + \mathrm{o}(1) \qquad \text{as } x \to \pm\infty \tag{2.5b}$$

for any finite y.

Throughout the paper $\Psi_G(\varphi, \boldsymbol{r_0})$ is the scattering diagram, $A_G^{\pm}(\boldsymbol{r_0})$ are amplitudes of the surface waves propagating in the positive/negative direction of the x-axis. The absence of limits of integration means that the contour of integration coincides with the real λ-axis except for small neighbourhoods of the points $\lambda = \pm k, \pm\kappa$. It avoids these points by following small semicircles with centres at these points. Those semicircles are located below $\lambda = k, \kappa$ and above $\lambda = -k, -\kappa$. The following functions are used in (2.1), (2.2)

$$l(\lambda) = (D\lambda^4 - m\omega^2)\gamma(\lambda) - \rho\omega^2 \qquad l_0(\lambda) = l(\lambda) + 2\rho\omega^2 \;\; \gamma(\lambda) = \sqrt{\lambda^2 - k^2}. \tag{2.6}$$

The function $\gamma(\lambda)$ is defined on the complex plane cut along the vertical lines $\lambda = k + \mathrm{i}t$ and $\lambda = -k - \mathrm{i}t$, $t \in [0, \infty)$, and $\gamma(\lambda) > 0$ for $\lambda > k$. The method of steepest descent [12] allows the following representations for the scattering diagram to be found

$$\Psi_G(\varphi, \boldsymbol{r_0}) = \frac{1}{4\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}kx_0\cos\varphi} \left(\mathrm{e}^{-\mathrm{i}ky_0\sin\varphi} + \frac{l_0(k\cos\varphi)}{l(k\cos\varphi)} \mathrm{e}^{\mathrm{i}ky_0\sin\varphi} \right) \tag{2.7a}$$

$$\Psi_g(\varphi, x_0) = -\frac{\rho\omega^2}{2\pi} \mathrm{e}^{-\mathrm{i}kx_0\cos\varphi} \frac{k\sin\varphi}{l(k\cos\varphi)}. \tag{2.7b}$$

Below other representations for Green functions $G(\boldsymbol{r}, \boldsymbol{r_0})$ and $g(\boldsymbol{r}, x_0)$ are needed. They arise from (2.1) and (2.2) by using the new variable

$$\lambda = k\cos\psi \tag{2.8}$$

$$G(\boldsymbol{r}, \boldsymbol{r_0}) = -\frac{1}{4\pi\mathrm{i}} \int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)} \mathrm{e}^{-\mathrm{i}kx_0\cos\varphi} \left(\mathrm{e}^{-\mathrm{i}ky_0\sin\psi} + \frac{l_0(k\cos\psi)}{l(k\cos\psi)} \mathrm{e}^{\mathrm{i}ky_0\sin\psi} \right) \mathrm{d}\psi \tag{2.9a}$$

$$g(\boldsymbol{r}, x_0) = \frac{\rho\omega^2}{2\pi} \int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)} \mathrm{e}^{-\mathrm{i}kx_0} \cos\psi \frac{k\sin\psi}{l(k\cos\psi)} \mathrm{d}\psi. \tag{2.9b}$$

Note that representation (2.9a) is given for $y > y_0$. The contour C coincides with the contour $(\pi - \mathrm{i}\infty, \pi) \cup (\pi, 0) \cup (0, \mathrm{i}\infty)$ except for small neighbourhoods of the points

$$\varphi_+ = \arccos\left(\frac{\kappa}{k}\right) = \mathrm{i}\ln\left(\frac{\kappa}{k} + \sqrt{\left(\frac{\kappa}{k}\right)^2 - 1}\right) \qquad \varphi_- = \arccos\left(-\frac{\kappa}{k}\right).$$

It avoids these points by following small semicircles with centres at these points. Those semicircles are located on the right-hand side of φ_+ and on the left-hand side of φ_-.

The elementary identity

$$g(\boldsymbol{r}, x_0) = -\frac{\partial G}{\partial y_0}(\boldsymbol{r}, x_0) \tag{2.10a}$$

implies

$$\Psi_g(\varphi, x_0) = -\frac{\partial \Psi_G}{\partial y_0}(\varphi, x_0, 0). \tag{2.10b}$$

The scattering diagrams $\Psi_G(\varphi, \boldsymbol{r_0})$ and $\Psi_g(\varphi, x_0)$ are defined for all (real) angles $\varphi \in [0, \pi]$. It is proved in [11] that the diagrams allow the analytic continuation for the complex φ-plane as meromorphic functions. The points $\psi = \varphi_\pm$ are the only poles located in a neighbourhood of the contour C. The direct comparison of the representations (2.7) and (2.9) implies the formulae

$$G(\boldsymbol{r},\boldsymbol{r_0}) = -\int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)}\Psi_G(\psi,\boldsymbol{r_0})\,\mathrm{d}\psi \qquad (2.11a)$$

$$g(\boldsymbol{r},x_0) = -\int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)}\Psi_g(\psi,x_0)\,\mathrm{d}\psi. \qquad (2.11b)$$

Formulae (2.11) represent Green functions in terms of analytic continuations of their scattering diagrams. It should be mentioned again that representation (2.11*a*) is proved for $y > y_0$ only (see the comment after (2.9)). Note that representations (2.9) and (2.11) satisfy identity (2.10*a*) and representations (2.7) satisfy identity (2.10*b*).

Two misprints in [11] should be mentioned here. The factor 1\i in (2.7*a*) is missed in the similar formula (2.7) of [11]. The minus sign in (2.10*a*) is missed in the similar formula (2.12) of [11]. Both factors were taken into consideration throughout [11] and were just missed when typing.

3. Sommerfeld's formula for the field scattered on a submerged plate with an arbitrary compact scatterer

Let Ω be a compact scatterer (a compact body with the boundary $\partial\Omega$, a finite set of cracks in the plate, a finite set of supporting ribs, etc). For example, let the scatterer Ω be a compact body attached to the plate and $\partial\Omega \cap \{y = 0\} = \{|x| = a, y = 0\}$. The boundary $\partial\Omega$ is supposed to be smooth and located above the line $y = 0$. Then $u(\boldsymbol{r}) = u(x, y)$ is the solution of the boundary value problem

$$(\Delta_2 + k^2)u(\boldsymbol{r}) = 0 \qquad \boldsymbol{r} \in \mathbb{R}^2_+\backslash\Omega \qquad (3.1)$$

$$\left(D\frac{\partial^4}{\partial x^4} - m\omega^2\right)\frac{\partial u}{\partial y}(x,0) + \rho\omega^2 u(x,0) = 0 \qquad a < |x| < \infty \qquad y = 0 \qquad (3.2)$$

$$u(\boldsymbol{r}) = f(\boldsymbol{r}) \qquad \boldsymbol{r} \in \partial\Omega \qquad (3.3)$$

subject to radiation conditions similar to (2.5)

$$u(\boldsymbol{r}) = \sqrt{\frac{2\pi}{kr}}\mathrm{e}^{\mathrm{i}(kr-\frac{\pi}{4})}\Psi_u(\varphi) + \mathrm{o}(r^{-\frac{1}{2}}) \qquad \text{as } r \to \infty \qquad (3.4a)$$

uniformly with respect to $\varphi \in [0, \pi]$, and

$$u(\boldsymbol{r}) = A_u^{\pm}\mathrm{e}^{\pm\mathrm{i}\kappa x - \sqrt{\kappa^2-k^2}y} + \mathrm{o}(1) \qquad \text{as } x \to \pm\infty \qquad (3.4b)$$

for any finite y. Let the plate be fixed at the points $\{x = \pm a, y = 0\}$, i.e.

$$\xi_u(\pm a) = \xi_u'(\pm a) = 0 \qquad (3.5)$$

where

$$\xi_u(x_0) = \frac{1}{\rho\omega^2}\frac{\partial u}{\partial y_0}(x_0, 0). \qquad (3.6)$$

Applying Green's second formula to the functions $u(\boldsymbol{r})$ and $G(\boldsymbol{r},\boldsymbol{r_0})$ in the domain $\{\mathbb{R}^2_+\backslash\Omega\} \cap \{r < R\}$ with sufficiently large R, using contact conditions (3.5), radiation conditions (3.4) and letting $R \to \infty$ yields the representation [13, 11]

$$u(\boldsymbol{r}) = \int_{\partial\Omega} A(\boldsymbol{r_0})G(\boldsymbol{r},\boldsymbol{r_0})\,\mathrm{d}s_0 + B(x_0)g(\boldsymbol{r},x_0)\Big|_{x_0=-a}^{a}. \qquad (3.7)$$

Here A and B are the following differential operators

$$A(\boldsymbol{r_0}) = f(\boldsymbol{r_0})\frac{\partial}{\partial \boldsymbol{n}_0} + \frac{\partial u}{\partial \boldsymbol{n}_0}(\boldsymbol{r}_0) \qquad B(x_0) = D\left(-\xi_u'''(x_0) + \xi_u''(x_0)\frac{\partial}{\partial x_0}\right) \tag{3.8}$$

and $\boldsymbol{n_0}$ is the unit normal vector to $\partial\Omega$. Representation (3.7) and the radiation conditions for the Green functions allows one to justify the radiation conditions (3.4) and to find the following expression for the scattering diagram

$$\Psi_u(\varphi) = \int_{\partial\Omega} A(\boldsymbol{r_0})\Psi_G(\varphi, \boldsymbol{r_0})\,\mathrm{d}s_0 + B(x_0)\Psi_g(\varphi, x_0)\Big|_{x_0=-a}^{a}. \tag{3.9}$$

The scattering diagrams in both the left- and right-hand side of identity (3.9) are defined on $\varphi \in (0, \pi)$. According to [11], they allow analytic continuation to the complex φ-plane as meromorphic functions with poles at the points at which $l(k\cos\varphi) = 0$. Hence, identity (3.9) is valid on the complex φ-plane except for those poles. Note that none of them are located on the contour C.

Substituting (2.11) into (3.7) and changing the order of integration yields for $y > \max\{y_0 | y_0 \in \partial\Omega\}$

$$u(\boldsymbol{r}) = \int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)}\left(\int_{\partial\Omega} A(\boldsymbol{r_0})\Psi_G(\psi, \boldsymbol{r_0})\,\mathrm{d}s_0 + B(x_0)\Psi_g(\psi, x_0)\Big|_{x_0=-a}^{a}\right)\mathrm{d}\psi. \tag{3.10}$$

Identities (3.9) and (3.10) imply

$$u(\boldsymbol{r}) = -\int_C \mathrm{e}^{\mathrm{i}kr\cos(\varphi-\psi)}\Psi_u(\psi)\,\mathrm{d}\psi. \tag{3.11}$$

Identity (3.11) represents the desired connection between the scattered field and the analytic continuation of the scattering diagram for this field. Note that this connection for Green functions is given by (2.11). Identity (3.11) is known for the scattering on a compact obstacle (if Dirichlet, Neumann or mixed boundary conditions on its surface are given) as Sommerfeld's formula [9, 10].

4. Uniqueness of the solution for the BCVP

Sommerfeld's formula allows the uniqueness of the solution for the boundary-contact value problem (3.1)–(3.6) to be proved.

If two different solutions of the problem exist, then their difference $u(\boldsymbol{r})$ solves the homogeneous problem. First, it can be shown that the scattering diagram and the amplitudes of the surface waves are not presented in such a solution. The proof is based on the application of the second Green formula to the solution $u(\boldsymbol{r})$ and its complex conjugate $\bar{u}(\boldsymbol{r})$ in the domain $\{\mathbb{R}^2_+ \backslash \Omega\} \cap \{r < R\}$ with sufficiently large R (see [13, 4]). Using conditions (3.2)–(3.6) and letting $R \to \infty$ yields the identity for the total scattered energy (see [14, 15])

$$E_{\mathrm{sc}} = \frac{\pi}{\rho\omega}\int_0^\pi |\Psi_u(\varphi)|^2\,\mathrm{d}\varphi + \frac{\kappa}{4\rho^2\omega^3}(5D\kappa^4 - 4Dk^2\kappa^2 - m\omega^2)(|A_u^+|^2 + |A_u^-|^2) = 0. \tag{4.1}$$

Therefore

$$\Psi_u(\varphi) = 0 \qquad 0 < \varphi < \pi \qquad \text{and} \qquad A_u^{\pm} = 0. \tag{4.2}$$

Formulae (4.2) do not mean that this solution is identically zero. Indeed, radiation conditions in this case only imply that

$$\begin{aligned} &u(\boldsymbol{r}) = \mathrm{o}(r^{-1/2}) \qquad \text{as } r \to \infty,\ \varphi \in (0, \pi) \\ &u(\boldsymbol{r}) = \mathrm{o}(1) \qquad \text{as } x \to \pm\infty. \end{aligned} \tag{4.3}$$

According to [11], the scattering diagram $\Psi_u(\varphi)$ is a meromorphic function on the complex φ-plane. In particular, it is analytic on the contour C described in section 2. According to (4.2), $\Psi_u(\varphi) = 0$ on the part of this contour, $0 < \varphi < \pi$. Hence, $\Psi_u(\varphi) = 0$ identically. Sommerfeld's formula (3.11) implies that

$$u(\boldsymbol{r}) = 0 \qquad \text{as } y > \max\{y_0 | y_0 \in \partial\Omega\}. \tag{4.4}$$

The analyticity of a solution for the Helmholtz equation allows us to state that $u(\boldsymbol{r}) = 0$ everywhere in $\{\mathbb{R}^2_+ \backslash \Omega\}$. Hence, the uniqueness of the solution for the BCVP (3.1)–(3.6) is proved.

The compact scatterer Ω was supposed to be located above the line $y = 0$. If this assumption is rejected, Sommerfeld's formula (3.11) is valid only for $y > \max\{0, \max_{y_0 \in \partial\Omega} y_0\}$. The proof of uniqueness of the solution for the BCVP has to be modified in this case. First, the proof of identities (4.2) does not use this assumption. Further, let Γ be an arbitrary smooth path between the points $\{x = -a, y = 0\}$ and $\{x = a, y = 0\}$ located above $\partial\Omega$ and the line $y = 0$. Sommerfeld's formula (3.11) is valid for the BCVP (3.1)–(3.6) with substitution $\partial\Omega$ for Γ. Hence, $u(\boldsymbol{r}) = 0$ in the domain with the boundary $\{|x| > a, y = 0\} \cup \Gamma$. The analyticity of the solution concludes the proof.

5. Conclusion: some other possible applications of Sommerfeld's formula

In many papers devoted to the scattering of acoustic waves on elastic plates with inhomogeneities the following representation for the scattered field (similar to (2.1), (2.2)) is used

$$u(\boldsymbol{r}) = \int p(\lambda) \mathrm{e}^{\mathrm{i}\lambda x - \gamma(\lambda) y} \, \mathrm{d}\lambda. \tag{5.1}$$

The function $p(\lambda)$ is unknown and should be found with the help of the boundary conditions. In the case of pointwise inhomogeneities, the function $p(\lambda)$ is represented explicitly in terms of some arbitrary constants, these constants are found by application of the contact conditions. In the case of extended inhomogeneities located on the plate $\{-\infty < x < \infty, y = 0\}$, the dual integral equations for $p(\lambda)$ arise, they are solved by application the Wiener–Hopf technique. Formula (5.1) represents the field as a sum of propagating plane waves (for $|\lambda| < k$) and inhomogeneous waves (for $|\lambda| > k$) with an unknown amplitude $p(\lambda)$. The following question arises. Does representation (5.1) contain all the possible solutions of the boundary-contact value problem? It can be easily checked that this representation arises from Sommerfeld's formula by substituting $\lambda = k \cos\psi$ with the function $p(\lambda) = -\Psi_u(\psi)/k \sin\psi$. Note that Sommerfeld's formula is proved above for $y > \max\{y_0 | y_0 \in \partial\Omega\}$. Hence, formula (5.1) represents the general form of the solution for a boundary-contact value problem is all inhomogeneities are located on the plate $\{-\infty < x < \infty, y = 0\}$. If some inhomogeneities are located above the plate, representation (5.1), generally speaking, is not valid up to the boundary $\{|x| > a, y = 0\} \cup \partial\Omega$. The conjecture on the possibility of an analytic continuation of Sommerfeld's formula up to the boundary of the domain is called the Rayleigh hypothesis (for a compact scatterer if Dirichlet, Neumann or mixed boundary conditions on its surface are given, see [16, 10]). The authors are currently working on an analysis of the Rayleigh hypothesis for BCVP.

Sommerfeld's formula (3.11) and analytical properties of the function $\Psi_u(\varphi)$ imply that $u(\boldsymbol{r}) = 0$ even if $\Psi_u(\varphi) = 0$ on any set of angles with a limiting point. This result might be of interest for inverse problems.

Kirchhoff's model of the plate is used in this paper. The authors' conjecture is that if the scatterer $\partial\Omega$ is an elastic shell and both the elastic plate and elastic shell are described by any correct theory, identity (3.11) is still valid.

The author was partially supported by The University of Tennessee at Chattanooga Center of Excellence in Computer Applications Scholarship.

References

[1] Crighton D G and Innes D 1983 Low frequency acoustic radiation and vibration response of locally excited fluid-loaded structures *J. Sound Vib.* **91** 293–314

[2] Cray B A 1994 Acoustic radiation from periodic and sectionally periodic rib-stiffened plates *J. Acoust. Soc. Am.* **95** 256–64

[3] Kousov D P 1967 On a solution of the Helmholtz equation for a half-plain with boundary conditions containing high order derivatives *J. Appl. Math. Mech.* **31** 164–70

[4] Belinskiy B P 1984 On the uniqueness of a solution for boundary value contact acoustic problems *Vestn. Leningr. Univ. Math.* **16** 139–46

[5] Belinskiy B P 1978 Sound radiation from a plate with protruding stiffener *Sov. Phys. Acoust.* **24** 183–7

[6] Andronov I V 1993 The propagation of sound in a fluid under an elastic plate with a crack *J. Appl. Math. Mech.* **57** 363–9

[7] Andronov I A and Belinskiy B P 1996 Acoustic scattering by a narrow crack in an elastic plate *Wave Motion* **24** 101–15

[8] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics 1: Functional Analysis* (New York: Academic)

[9] Sommerfeld A 1950 *Vorlesungen Über Theoretische Physick* vol VI (Wiesbaden)

[10] Apelcin V F and Kyurkchan A G 1990 *Analiticheskie Svojstva Volnovyh Polej* (Moscow: MGU) (in Russian)

[11] Andronov I V, Belinskiy B P and Dauer J P 1996 The connection between the scattering diagram and the amplitudes of the surface waves for acoustic scattering by a baffled flexible plate *J. Sound Vib.* **195** 667–73

[12] Erdélyi A 1956 *Asymptotic Expansions* (New York: Dover)

[13] Belinskiy B P and Kousov D P 1981 Green's type formulas for flexurally vibrating plates *Acoustic J.* **27** 394–8 (in Russian)
See also Belinskiy B P and Kousov D P 1981 *Sov. Phys. Acoust.* **27** 394–8

[14] Belinskiy B P and Kousov D P 1980 The optical theorem for the plate-liquid system *Acoustic J.* **26** 13–19 (in Russian)
See also Belinskiy B P and Kousov D P 1980 *Sov. Phys. Acoust.* **26** 8–11

[15] Belinskiy B P 1990 Comments on 'An optical theorem for acoustic scattering by baffled flexible surfaces' *J. Sound Vib.* **139** 522–3

[16] Lord Rayleigh 1907 *Proc. R. Soc.* A **79** 399–416